MOTION OF A THIN LAYER OF A HEAVY FLUID OVER THE SURFACE OF A SOLID BODY

(O DVIZHENII TONKOGO SLOIA TIAZHELOI ZHIDKOSTI PO POVERKHNOSTI TVERDOGO TELA)

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1. Potential flow of an ideal weightless fluid, spreading in a thin layer over the surface of a solid body, was considered in [1 and 2]. The equations of a plane flow of a slightly curved jet of a heavy fluid of the type considered here have the form [3 :

$$v_s \frac{\partial v_s}{\partial s} + v_r \frac{\partial v_s}{\partial r} = - \frac{1}{\rho} \frac{\partial p}{\partial s} - g \frac{dy}{ds}, \qquad -k(s) v_s^2 = - \frac{1}{\rho} \frac{\partial p}{\partial r} + g \frac{dx}{ds}$$

$$\frac{\partial v_s}{\partial s} + \frac{\partial v_r}{\partial r} = 0 \qquad (1.1)$$

Here v_s, v_r are the components of the velocity of the fluid, measured along the axes of an auxiliary curvilinear system of coordinates s, r, in which r lies along a normal to the surface of the solid body, and s is the length of the arc of the directrix L_1 of the cylinrical surface of the solid body, p and ρ are, respectively, the pressure and the density of the fluid, g is the acceleration of gravity, directed against the y-axis of the basic orthogonal system of coordinates $x = x(s)$ and $y = y(s)$ are the equations of L_1, and $k(s)$ is the curvature of L_1.

Transferring the origin of r to the free stream line L_0, we have for $r = 0$

$$v_s = v_0(s), \qquad v_r = 0, \qquad p = p_0 = \text{const} \qquad (1.2)$$

On the basis of the Bernoulli integral for a steady motion, we have $v_0^2(s) = c_0 - 2gy(s)$. We shall regard the thickness of the stream $h(s)$, measured along the normal to the surface of the solid body, as a small quantity of the first order.

In view of this, velocity v_s can be expressed in the form

$$v_s(s, r) = v_0(s) + u(s, r) \qquad (1.3)$$

where $u(s,r)$ is a quantity of the first order of magnitude. Substituting (1.3) into the second equation of syseem (1.1) and bearing in mind that $k(s)$ is also a small quantity of the first order, we ignore the terms of the second and higher orders,

$$k(s) v_0^2(s) + g \frac{dx}{ds} = \frac{1}{\rho} \frac{\partial p}{\partial r}$$

From this, taking into account conditions (1.2), we obtain the pressure in the stream

$$p = \rho\left[k(s)\, v_0^2(s) + g\,\frac{dx}{ds}\right] r + p_0 \qquad (1.4)$$

Using this equation to eliminate pressure p in the first equation of system (1.1) and introducing the usual stream function $\psi(s,r)$ defined by

$$v_s = \frac{\partial\psi}{\partial r}, \qquad v_r = -\frac{\partial\psi}{\partial s}$$

we obtain the following nonlinear equation for the stress function

$$\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial r\,\partial s} - \frac{\partial\psi}{\partial s}\frac{\partial^2\psi}{\partial r^2} + \frac{d}{ds}\left[k(s)\, v_0^2(s) + g\,\frac{dx}{ds}\right] r + g\,\frac{dy}{ds} = 0 \qquad (1.5)$$

By reference to (1.3), the stream function $\psi(s,r)$ can be expressed in the form

$$\psi(s,r) = \psi_0(s,r) + \psi_1(s,r) \qquad (1.6)$$

where $\psi_0 = rv_0(s)$, and function $\psi_1(s,r)$ is one order of magnitude smaller than $\psi_0(s,r)$. Ignoring in Equation (1.5) all terms involving the products of derivatives of function $\psi_1(s,r)$ we obtain the linear equation

$$v_0(s)\frac{\partial^2\psi_1}{\partial r\,\partial s} + \frac{dv_0}{ds}\frac{\partial\psi_1}{\partial r} - r\,\frac{dv_0}{ds}\frac{\partial^2\psi_1}{\partial r^2} + r\,\frac{d}{ds}\left[k(s)\, v_0^2(s) + g\,\frac{dx}{ds}\right] = 0 \qquad (1.7)$$

The solution of this equation, satisfying also the conditions of (1.2), is

$$\psi_1(s,r) = r^2 f(s), \qquad \frac{df}{ds} = -\frac{1}{2v_0(s)}\frac{d}{ds}\left[k(s)\, v_0^2(s) + g\,\frac{dx}{ds}\right] \qquad (1.8)$$

Thus, the stream function of a slightly curved jet of a heavy fluid (to the present degree of accuracy) is

$$\psi(s,r) = rv_0(s) - r^2\left(\int \frac{d}{ds}\left[v_0^2(s)\, k(s) + g\,\frac{dx}{ds}\right]\frac{ds}{2v_0(s)} + c_1\right) \qquad (1.9)$$

To determine the constant c_1 it is necessary to specify the velocities v_s and v_r and the pressure p at the initial section $s = s_0$ in the form

$$v_r = \alpha r + \beta r^2, \qquad v_s = v_0(s_0) + \delta r, \qquad p = p_0 + \lambda r \qquad (\alpha, \beta, \delta, \lambda - \text{const})$$

where α, β, λ are explicitly defined in terms of the geometric characteristics of the surface over which the flow takes place at $s = s_0$ and $\delta = c_1$. The thickness of the jet is determined from the condition $\psi(s,h) = Q$, where Q is the rate of flow, since L_1 is the streamline.

2. We shall now use these relationships to determine the shape of the jet of a heavy fluid on the basis of the given distribution of pressure p_1 along the streamline L_1. etermining the thickness of the jet $h(s)$ to the first approximation [1] on the basis of the velocity $v_0(s)$, we obtain from (1.4) the following equation for the shape of the jet:

$$k(s)\, v_0^2(s) + g\,\frac{dx}{ds} = \frac{v_0(s)\,\Delta p}{Q} \qquad (\Delta p = p_1 - p_0) \qquad (2.1)$$

Let p_1 be a known function $y(s)$, that is $p_1 = p_1(y)$. It is known [4] that

$$k(s) = \frac{d^2y}{ds^2}\left[1 - \left(\frac{dy}{ds}\right)^2\right]^{-1/2}, \qquad dx^2 + dy^2 = ds^2 \qquad (2.2)$$

Therefore, stipulating that $dy/ds = B(y)$, we obtain from (2.1)

$$-(c_0 - 2gy)\frac{d}{dy}\sqrt{1 - B^2} + g\sqrt{1 - B^2} = \frac{\Delta p(y)}{Q}\sqrt{c_0 - 2gy} \qquad (2.3)$$

Whence,

$$1-\left(\frac{dy}{ds}\right)^2=\frac{J^2(y)}{c_0-2gy} \qquad \left(J(y)=\int\frac{\Delta p(y)}{Q}\,dy+c_2\right) \tag{2.4}$$

Using (2.2) to eliminate the length of the arc, we shall obtain the following double quadrature which we can use to define the equation of the trajectory of the jet:

$$x=\int\frac{J(y)\,dy}{[c_0-2gy-J^2(y)]^{1/2}}+c_3 \tag{2.5}$$

The constants c_2 and c_3 are determined from the initial point $y(x_0)$ and the initial slope $y'(x_0)$ of the jet. Let us now consider some particural cases.

a) Let $\Delta p=0$, $p_1=p_0$. The trajectory of the free falling jet will be a parabola

$$2gy=(c_0-c_2^2)-\frac{4g^2(x-c_3)^2}{c_2^2}$$

b) In the case of impact of a jet of heavy fluid against the surface of a static heavy fluid with spwcific gravity γ, when at the interface of the jet and the static fluid there is a discontinuity in the velocities, and the pressure can be assumed equal to the hydrostatic pressure. The trajectory of the jet will be defined by the quadrature

$$x=\int\frac{(^1/_2\gamma Q^{-1}y^2+c_2)\,dy}{[c_0-2gy-(^1/_2\,\gamma Q^{-1}y^2+c_2)^2]^{1/2}}+c_3 \tag{2.6}$$

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